Published by Institute of Physics Publishing for SISSA

Received: *June 25, 2008*
Accepted: *July 22, 2008*
Published: *August 13, 2008*

$\mathcal{N} = 4$ SYM on S^3 with near critical chemical potentials

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Abstract: We study the $\mathcal{N} = 4$ theory at weak coupling, on a three sphere in the grand canonical ensemble with R symmetry chemical potentials. We focus attention on near critical values for the chemical potentials, above which the classical theory has no ground state. By computing a one loop effective potential for the light degrees of freedom in this regime, we show the existence of flat directions of complex dimension N, $2N$ and $3N$ for one, two and three critical chemical potentials respectively; these correspond to one half, one quarter and one-eighth BPS states becoming light respectively at the critical values. At small finite temperature we show that the chemical potentials can be continued beyond their classical limiting values to yield a deconfined metastable phase with lifetime diverging in the large N limit. Our low temperaure analysis complements the high temperature metastability found by Yamada and Yaffe. The resulting phase diagram at weak coupling bears a striking resemblance to the strong coupling phase diagram for charged AdS black holes. Our analysis also reveals subtle qualitative differences between the two regimes.

Keywords: AdS-CFT Correspondence, Thermal Field Theory.

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1. Introduction

Among the many deep insights to emerge from the AdS/CFT correspondence [1 – 3], is the remarkable connection between large N Yang-Mills thermodynamics at strong coupling and black holes in AdS spacetimes. A dramatic consequence of this equivalence is that the first order Hawking-Page transition in string theory on asymptotically $AdS_5 \times S^5$ geometries corresponds to a deconfinement phase transition in the $\mathcal{N} = 4$ supersymmetric, large N Yang-Mills theory on the conformal boundary of the spacetime, namely on $S^3 \times S^1$ [4]. Although the above picture most naturally applies at strong 't Hooft coupling, it is now well appreciated that such a first order deconfinement transition occurs even in the large N *free* Yang-Mills theory on $S^3 \times S^1$, with thermal boundary conditions [5 – 7]. It is believed that the behaviour might possibly extend to the weakly interacting theory.

This raises the exciting possibility that thermodynamics of the weakly coupled $\mathcal{N} = 4$ theory on S^3 could be qualitatively similar to the strongly interacting case and may provide a window into the physics of black holes in string theory. The outstanding issue is to understand how the regimes of weak and strong 't Hooft coupling are mapped into one another as the coupling is changed. Of course, it is still plausible that, away from the extreme limits of infinite and zero 't Hooft couplings, the phase structure at non-zero weak

coupling could be qualitatively different from that at finite strong coupling. This latter possibility, if true, would imply non-analyticities in the theory as a function of the 't Hooft coupling, and qualitatively different thermodynamics of string theory on strongly curved backgrounds [8, 9] compared to semiclassical gravity.

For these reasons, mapping out the thermodynamic phase structure of $\mathcal{N} = 4$ theory has received much attention from various viewpoints in recent years [10 – 21].

In this paper we study the weakly coupled $\mathcal{N} = 4$ theory with SU(N) gauge group, on an S^3 of radius R, in the grand canonical ensemble with chemical potentials for global R-symmetry charges. The $\mathcal{N} = 4$ theory has three global U(1) symmetries generated by the Cartan elements of the SU(4) R-symmetry. To pass to the grand canonical ensemble, we introduce chemical potentials (μ_1, μ_2, μ_3) for the three global R-symmetry charges (J_1, J_2, J_3) in the theory.

We will explore a particularly interesting corner of the phase diagram which turns out to be analytically tractable in the presence of chemical potentials. When the $\mathcal{N} = 4$ theory is formulated on the three sphere of radius R, the six scalar fields obtain a mass $1/R$ due to conformal coupling to the curvature of the sphere. In addition, the introduction of a chemical potential for a global charge induces an effective negative mass squared for scalars carrying that charge. There is thus a critical regime of values for the chemical potentials $\mu_p = R^{-1} + \mathcal{O}(\lambda)$, for which the scalars become light degrees of freedom. (Here $\lambda \ll 1$ is the 't Hooft coupling.) We will refer to this region as the "near critical" regime.

The classical theory is only stable when $\mu_p \leq R^{-1}$, and this statement is true also at one loop order. When the chemical potential exceeds this value, the classical Hamiltonian becomes unbounded from below. Note that in a simpler field theory, such as a massive complex scalar field theory with a quartic interaction, introducing a chemical potential exceeding the mass leads to Bose-Einstein condensation. This does not appear to be the case, at least perturbatively, in the $\mathcal{N} = 4$ theory where the quantum corrections at weak coupling are always systematically smaller than the tree level mass term. The only situation where there is a possibility for tree level and one loop radiative corrections to compete is when $\mu_p \simeq R^{-1} + \mathcal{O}(\lambda)$, so that the classical potential already contains a term of order λ.

In addition to requiring near-criticality of chemical potentials, for the most part we will focus on low temperatures $TR \ll 1$. Our analysis complements the work of Yamada and Yaffe [11] who investigated the weakly interacting theory[1] in the temperature range $1 \ll TR \lesssim 1/\sqrt{\lambda}$. One of their interesting results was to demonstrate the existence of a metastable deconfined plasma phase, at high temperatures in the range $R^{-1} \leq \mu_p < \sqrt{\lambda T^2 + R^{-2}}$.

In the near critical regime where two or more scalars are light, there exist classically (almost) flat directions. They are parametrized by mutually commuting, constant background values for the light scalars. At a small finite temperature, the shallow directions include mutually commuting constant configurations for the thermal Wilson-Polyakov line. Moving along these directions generically Higgses the theory: SU(N) $\to$ U(1)$^{N-1}$. For

[1] The authors of [11] also studied the *free theory* in the grand canonical ensemble, obtaining a line of first order Hagedorn/deconfinement transitions for generic values of the chemical potential. We do not address the issue as to whether or not the first order line gets modified at non-zero weak coupling.

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sufficiently large values of the diagonal modes of the scalars, the off-diagonal modes of all the Kaluza-Klein harmonics acquire large masses. These can be integrated out to generate a Wilsonian effective potential at one loop for the light, diagonal modes in the regime of near critical chemical potentials.

Our first result in the near critical regime at $T = 0$ is that the one loop radiative correction to the scalar potential vanishes, following a non-trivial Bose-Fermi cancellation in regularized Casimir sums in the presence of expectation values for the light scalar modes. This means that the nearly flat directions (at $T = 0$) are not lifted by quantum corrections in the vicinity of critical chemical potentials.

In particular, when the chemical potentials are *at* their critical values, and at $T = 0$, due to vanishing quantum corrections there is a Coulomb branch moduli space of complex dimension N, $2N$ or $3N$, depending on whether we have one, two or three critical chemical potentials switched on. For each case we also have a different number of zero energy modes: With $\mu_1 = R^{-1}, \mu_2 = \mu_3 = 0$ a single *holomorphic* adjoint scalar mode with zero energy appears, while for $\mu_1 = \mu_2 = R^{-1}, \mu_3 = 0$, there are 2 holomorphic scalar zero modes. The situation with three critical chemical potentials reveals two adjoint fermion zero energy modes along with 3 holomorphic scalar zero modes.

The appearance of the moduli spaces at critical chemical potential and the associated zero energy modes can be understood more generally as follows. The generator of time translations of the $\mathcal{N} = 4$ theory with chemical potentials on S^3, may be expressed as

$$\Delta(\mu_p) = \Delta - \sum_{p=1}^{3} \mu_p J_p, \tag{1.1}$$

where Δ is identified with the dilatation operator of the theory formulated on $\mathbb{R}^4$. With one critical chemical potential $\mu_1 = R^{-1}$ (and $\mu_2 = \mu_3 = 0$), this operator vanishes on all states with $R\Delta = J_1$, which are the infinite set of $\frac{1}{2}$ BPS operators in the theory. At the critical values for μ_p, the superconformal algebra ensures positive definiteness of the above Hamiltonian and the $\frac{1}{2}$ BPS operators thus constitute the infinitely degenerate set of ground states of the theory. This can be interpreted as the origin of the flat directions at critical chemical potential. For two and three critical chemical potentials, the ground states are parametrized by $\frac{1}{4}^{\text{th}}$ and $\frac{1}{8}^{\text{th}}$ BPS operators respectively. The dimensions of the moduli spaces we find and associated zero modes are consistent with expectations based on our knowledge of the generators of $\frac{1}{2}$ BPS, $\frac{1}{4}^{\text{th}}$ BPS and $\frac{1}{8}^{\text{th}}$ BPS states in the $\mathcal{N} = 4$ theory [22, 23].

At the critical values for the chemical potentials, upon switching on a small non-zero temperature $TR \ll 1$ the theory acquires another set of zero modes in addition to the light scalars. These new zero modes are the diagonal elements of the Polyakov loop matrix. We find a joint effective potential for all the light modes and deduce that eigenvalues of the Polyakov loop matrix experience purely a mutual attractive force causing them to all collapse on to a point. This corresponds to a deconfined phase wherein the trace of the Polyakov loop has non-zero expecatation value. Furthermore, the theory develops a mass gap due to exponentially small thermal masses at low temperature. This means

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that thermal effective potential for the scalars has a positive curvature near the origin at critical chemical potential. Away from the origin, at large field amplitudes it asymptotes to a constant $(3/16R)$. The small positve curvature near the origin allows us to raise the chemical potentials beyond their classical limiting value $1/R$, and still obtain a locally stable configuration at the origin. Raising the chemical potential(s) above $1/R$, however, causes the scalar potential to have a runaway behaviour (unbounded from below) at large field amplitudes. This results in a metastable state with an exponentially diverging lifetime in the $N \to \infty$ limit. We expect this low temperature metastable phase to be a smooth continuation of the high temperature metastable plasma discovered in [11].

Technically, there is an important difference between the high and low temperature regimes. At high temperatures, $TR \gg 1$, the effective potential is obtained basically by a flat space computation on $\mathbb{R}^3$. The low temperature effective potential on the other hand, in the presence of non zero R-charge densities ($\mu_p \neq 0$) depends on the details of the compact space on which the theory is formulated. Our analyisis complements the work of [11], filling in the low temperature regime of the phase diagram of the theory with chemical potentials. The final weak coupling phase diagram is shown in figure 3.

Perhaps surprisingly, a quick comparison of figure 3 with the strong coupling phase diagram in figure 4 reveals striking similarities. At strong coupling, the phase diagram is dictated by the thermodynamics and stability properties of R-charged black holes in AdS space. More detailed aspects of these are discussed in section 4. Here we further remark that it has also recently been found [24] that the region in the strong coupling phase diagram, below the black hole instability line and above the critical value of the chemical potential, exhibits a metastability . This metastability corresponds to a singe (probe) D3-brane splitting from a cluster of large N rotating branes whose near horizon geometry is the charged AdS black hole background.[2] This is exactly the physics expected from the weak coupling analysis where the lifetime of the metastable state is determined by the probability for one scalar eigenvalue to tunnel out or be thermally activated into the unstable region.

Our weak coupling analysis, however, also reveals certain important differences with the strongly coupled regime. We find that, in the $\mu - T$ plane, the metastable region shrinks to zero size at zero temperature — the instability line meets the first order deconfinement line at $\mu = 1/R$ and $T = 0$ for any choice of chemical potentials. At strong coupling, the black hole instability line and the first order Hawking-Page lines meet at $\mu = 1/R$ and non-zero temperature. Only for the case with equal chemical potentials do the two lines meet at $T = 0$ and $\mu = 1/R$ at strong coupling.

The organization of this paper is as follows. In section 2, we review how R-symmetry chemical potentials are introduced in the $\mathcal{N} = 4$ theory. In section 3, we show how to compute the one loop effective potential for the light degrees of freedom near critical values for chemical potentials. We perform the calculations with one, two and three critical chemical potentials at zero temperature. In sections 3.2 and 3.3 we establish the existence of

[2] As emphasized in [24] this phenomenon is distinct from the source of local thermodynamic instability found in [25, 26].

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flat directions at zero temperature and their interpretation in terms of BPS states becoming light. Sections 3.4 - 3.5 are devoted to establishing the existence of the metastable plasma phase at low temperatures. In section 4, we describe the similarities and differences between the phase diagrams at weak and strong coupling. Conclusions and future directions are summarized in section 5, and finally, an appendix is devoted to the spherical harmonic decomposition of the theory on S^3.

2. R symmetry chemical potentials

The $\mathcal{N} = 4$ theory has an $\mathrm{SU}(4)_R$ global symmetry. There are thus three chemical potentials which can be introduced associated to the maximal abelian subgroup $\mathrm{U}(1)^3 \subset \mathrm{SU}(4)_R$. The six adjoint scalars $\{\phi_i\}$ $(i = 1, 2, \ldots, 6)$ transform as the antisymmetric **6** of $\mathrm{SU}(4)_R$, while the fermions are in the fundamental representation, the **4** of $\mathrm{SU}(4)_R$. (We follow the conventions of [11] below).

We choose the three $\mathrm{U}(1)$ generators of the Cartan subalgebra to be

$$\begin{aligned} R_1^{\mathbf{4}} &= \frac{1}{2}\,\mathrm{diag}(1, 1, -1, -1), \\ R_2^{\mathbf{4}} &= \frac{1}{2}\,\mathrm{diag}(1, -1, 1, -1), \\ R_3^{\mathbf{4}} &= \frac{1}{2}\,\mathrm{diag}(1, -1, -1, 1), \end{aligned} \tag{2.1}$$

in the fundamental representation. Packaging the six real scalars into three complex combinations,

$$\Phi_1 = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \Phi_2 = \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4), \quad \Phi_3 = \frac{1}{\sqrt{2}}(\phi_5 + i\phi_6) \tag{2.2}$$

we can define the six-vector

$$\vec{\Phi} = (\Phi_1, \Phi_1^*, \Phi_2, \Phi_2^*, \Phi_3, \Phi_3^*). \tag{2.3}$$

The three $\mathrm{U}(1)$ generators acting in this representation are then

$$\begin{aligned} R_1^{\mathbf{6}} &= \frac{1}{2}\,\mathrm{diag}(1, -1, 0, 0, 0, 0), \\ R_2^{\mathbf{6}} &= \frac{1}{2}\,\mathrm{diag}(0, 0, 1, -1, 0, 0), \\ R_3^{\mathbf{6}} &= \frac{1}{2}\,\mathrm{diag}(0, 0, 0, 0, 1, -1), \end{aligned} \tag{2.4}$$

which clearly assigns opposite charges to the fields Φ_i and their complex conjugates.

The grand canonical partition function is defined to be

$$\mathcal{Z}(T, \mu_p) = \mathrm{Tr}\, e^{-\beta\left(\Delta - \sum_p \mu_p J_p\right)}\ , \tag{2.5}$$

where the J_p are the three associated conserved R charges. The charge densities involve both fermionic and bosonic contributions. With the above choice of the $\mathrm{U}(1)$ generators,

the chemical potential assignments are determined via the following expressions

$$\sum_{i=1}^{3} \mu_i \, R_i^{\mathbf{6}} = (\mu_1, -\mu_1, \mu_2, -\mu_2, \mu_3, -\mu_3) \tag{2.6}$$

for the six scalars in eq.(2.3). Similarly, the fermion chemical potentials are determined as

$$\sum_{i=1}^{3} \mu_i \, R_i^{\mathbf{4}} = \text{ diag}(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4) \tag{2.7}$$

where

$$\begin{aligned} \bar{\mu}_1 &= \frac{1}{2}(\mu_1 + \mu_2 + \mu_3) \\ \bar{\mu}_2 &= \frac{1}{2}(\mu_1 - \mu_2 - \mu_3) \\ \bar{\mu}_3 &= \frac{1}{2}(-\mu_1 - \mu_2 + \mu_3) \\ \bar{\mu}_4 &= \frac{1}{2}(-\mu_1 + \mu_2 - \mu_3)\,. \end{aligned} \tag{2.8}$$

The grand canonical partition function eq.(2.5) can also be realized as a Euclidean functional integral for the theory on $S^3 \times S^1$. In the functional integral or Lagrangean formulation, the chemical potential for each global charge can be thought of as introducing by hand, a constant (imaginary) background for the time component of a gauge field associated to the respective global U(1) symmetry. In the presence of the R charge chemical potentials the Lagrangean for the $\mathcal{N} = 4$ theory on $S^3 \times S^1$ becomes

$$\begin{aligned} \mathcal{L} = \frac{1}{g^2}\text{Tr}\Bigg(&\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sum_{p=1}^{3}(D_\nu\phi_{2p-1} - i\mu_p\delta_{\nu,0}\phi_{2p})^2 + \frac{1}{2}\sum_{p=1}^{3}(D_\nu\phi_{2p} + i\mu_p\delta_{\nu,0}\phi_{2p-1})^2 \\ &+ \frac{1}{2R^2}\phi_a^2 - \frac{1}{2}[\phi_a, \phi_b]^2 + i\bar{\psi}_A\left(\not{D} - \bar{\mu}_A\gamma_0\gamma_5\right)\psi_A - \bar{\psi}_A\left[\alpha^p_{AB}\phi_{2p-1} + i\beta^p_{AB}\gamma_5\phi_{2p},\, \psi_B\right]\Bigg) \end{aligned} \tag{2.9}$$

where all derivatives are gauge covariant derivatives on S^3. The ψ^A are four component Majorana fermions

$$\psi^A = \left(\lambda^A_\alpha,\, \bar{\lambda}^{A\dot{\alpha}}\right)^T \tag{2.10}$$

The indices $a, b = 1, \ldots 6$, and $A, B = 1, \ldots 4$, while $p = 1, 2, 3$. The 4×4 matrices, α^p and β^p are the Clebsch-Gordan coefficients, satisfying

$$\{\alpha^p, \alpha^q\} = -2\delta^{pq}, \quad \{\beta^p, \beta^q\} = -2\delta^{pq}, \quad [\alpha^p, \beta^q] = 0. \tag{2.11}$$

Explicit representations for the α and β matrices are given in terms of Pauli matrices as,

$$\alpha^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \qquad \alpha^2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \alpha^3 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \tag{2.12a}$$

$$\beta^1 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \qquad \beta^2 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \qquad \alpha^3 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \tag{2.12b}$$

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Classical (in)stability. It is immediately clear from the scalar kinetic terms in (2.9) that in the presence of chemical potentials the Euclidean action is not real. Notice also that with the chemical potential, the conformal scalars have an effective mass given by

$$m_p^2 = R^{-2} - \mu_p^2 \qquad p = 1, 2, 3. \tag{2.13}$$

Hence only when $m_p^2 \geq 0$, does the classical theory have a stable vacuum. For $\mu_p > R^{-1}$ there is a classical instability in the theory along directions in field space for which ϕ_{2p-1} and ϕ_{2p} commute. It follows that in flat space, i.e. on $\mathbb{R}^4$, where all the fields are exactly massless, it is not possible to introduce a chemical potential in $\mathcal{N} = 4$ SYM since the resulting theory has no ground state (at least classically) and the grand canonical ensemble is ill-defined. In finite volume and in particular on S^3, the conformal coupling of the scalars to the background curvature allows for a mass which leads to a stable vacuum for a range of values of the chemical potential.

It is worth noting that this kind of instability in a massive *interacting* scalar field theory, driven by a chemical potential exceeding the mass, generally leads to Bose-Einstein condensation, i.e. a VEV for the scalar fields and a spontaneous breaking of the U(1) symmetry. For example, this occurs in the ϕ^4 theory, where the interactions stabilize the ground state at some finite VEV. In the $\mathcal{N} = 4$ theory, classically at least there is nothing to stabilize the theory along the mutually commuting directions of configuration space when one or more of the μ_p exceed R^{-1}. Physically, when $\mu_p > R^{-1}$ the system can always lower the energy by populating the vacuum with any number of charged quanta which it can borrow from the bath.

In the vicinity of the critical chemical potentials $\mu_p \sim R^{-1}$, the scalar fields of the theory are the light, almost massless degrees of freedom. The appearance of these new light modes makes the approach to the critical chemical potential an interesting regime to study. In this paper we consider these near-critical regions in more detail. There are three cases which will each be considered separately:

$$\begin{array}{lll} \text{(i)} & \mu_1 \simeq R^{-1}, & \mu_2 = \mu_3 = 0; \\ \text{(ii)} & \mu_1 \simeq \mu_2 \simeq R^{-1}, & \mu_3 = 0; \\ \text{(iii)} & \mu_1 \simeq \mu_2 \simeq \mu_3 \simeq R^{-1}. & \end{array} \tag{2.14}$$

We will consider the approach to the critical chemical, with vanishing as well as small non-zero temperatures.

3. One loop effective potential — generalities

In this section of the paper, we set up the calculation of a one-loop effective potential for the $\mathcal{N} = 4$ theory on S^3 with a radius R, at both zero and non-zero (but small) temperatures in each of the near-critical regions (2.14) above.

Adopting a Wilsonian approach, we will compute the effective potential for the lightest degrees of freedom in theory, by integrating out all the other heavy modes in the background of the light modes. The natural mass scales for most of the heavy degrees of freedom are

the inverse radius of the S^3, namely R^{-1} and the temperature $T = \beta^{-1}$. All the fields can be expanded in terms of spherical harmonics on S^3 and Matsubara modes on the thermal circle. The light modes must necessarily be constant modes on $S^3 \times S^1$. The only fields which have such constant modes are the scalars ϕ_a and A_0, the gauge field component around S^1.[3] The scalars generally have a mass of order R^{-1} since they are conformally coupled to the background curvature. However, as we have seen above, in the presence of a near-critical chemical potential, $\mu_p \simeq R^{-1}$, the effective masses of ϕ_{2p-1} and ϕ_{2p} are small and so we should include their constant modes in the effective potential.

Thus there are two sets of light modes in the near critical theory. First, we have for each near-critical μ_p, light adjoint scalar modes which are the homogeneous parts of the respective fields on S^3:

$$\varphi_{2p-1} = \frac{T}{2\pi^2 R^3}\int_{S^3\times S^1}\phi_{2p-1}\,, \qquad \varphi_{2p} = \frac{T}{2\pi^2 R^3}\int_{S^3\times S^1}\phi_{2p}\,; \qquad \mu_p \simeq R^{-1}. \tag{3.1}$$

In addition to these, at finite temperature, we must also account for the spatial zero mode of the holonomy of the time component of the gauge field around the thermal circle:

$$\alpha = \frac{T}{2\pi^2 R^3}\int_{S^3\times S^1} A_0. \tag{3.2}$$

The Wilsonian effective potential has a tree-level contribution, as well as loop corrections,

$$V_{\text{eff}} = \lambda^{-1}V_0 + V_1 + \mathcal{O}(\lambda). \tag{3.3}$$

Here $\lambda = g^2 N$ is the 't Hooft coupling. Although we only restrict to one loop computations, we will consistently express all quantities as fucntions of the 't Hooft coupling and N, indpendently, since these are the parameters relevant for evenetual comparison with the large N gravity dual. The tree-level term is

$$V_0 = \frac{N\pi^2 R^3}{2\lambda}\text{Tr}\left(-\sum_a [A_0,\phi_a]^2 - \sum_{a<b}[\phi_a,\phi_b]^2 + \sum_p (R^{-2}-\mu_p^2)(\phi_{2p-1}^2+\phi_{2p}^2)\right)\,. \tag{3.4}$$

When the chemical potentials are small so that $\mu_p < R^{-1}$, the tree-level potential forces the associated scalar VEVs to vanish. On the other hand, if $\mu_p > R^{-1}$, the theory becomes unstable. In the region of one or more near-critical chemical potentials, $\mu_p \simeq R^{-1}$, the classical potential has almost flat directions. These flat directions correspond to mutually commuting VEVs φ_{2p-1} and φ_{2p}, in addition to a mutually commuting α.

One loop correction near-critical chemical potentials

We will now outline the computation of the one-loop effective potential in the theory approaching the critical chemical potential in each of the three regions (2.14). The near-critical region for the chemical potentials will be defined specifically as

$$\mu_p = R^{-1} + \mathcal{O}(\lambda). \tag{3.5}$$

[3]The other gauge field components A_i are vector-valued on S^3 and so do not have a constant mode.

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This parametrically small approach towards criticality is chosen so that the tree-level potential for the commuting modes is of the same order as the first loop correction. It is in this situation that there is the possibility of competition between these two effects with potentially interesting physics. If $|\mu_p - R^{-1}|$ is parametrically larger, then depending on the sign, the theory will simply either have a stable vacuum at the origin, or have perturbatively unstable runaway directions.

For each critical chemical potential μ_p, classically the theory will have almost flat directions parametrized by (a holomorphic combination of) the diagonal elements of mutually commuting matrices φ_{2p-1} and φ_{2p}, $(p = 1, 2, 3)$. In addition, there is another set of moduli parametrized by the diagonal elements of α, which also commutes with φ_{2p} and φ_{2p-1} at a minimum of the classical potential. Therefore at a generic point along this flat potential we have,

$$\alpha = \mathrm{diag}(\alpha_i) \quad \varphi_{2p-1} = \mathrm{diag}\,(\varphi_{2p-1\,i})\,, \quad \varphi_{2p} = \mathrm{diag}\,(\varphi_{2p\,i})\,, \quad i = 1, 2, \ldots, N. \tag{3.6}$$

Recall from (3.1) that the fields (α, φ_a) are the background values for the spatially homogeneous parts of the full quantum fields (A_0, ϕ_a).

Near-critical chemical potential for only one of the three U(1)s

We begin our analysis with the simplest situation where only one near critical chemical potential μ_1 is turned on:

$$\mu_1 \simeq R^{-1}, \mu_2 = \mu_3 = 0. \tag{3.7}$$

In this situation, the diagonal elements of the fields φ_1, φ_2 and α are the lightest modes in the theory.

At generic points of the classically flat directions parametrized by the diagonal elements of φ_1 and φ_2, we will integrate out all the inhomogeneous modes (the Kaluza-Klein harmonics on S^3) as well as the off-diagonal homogeneous fluctuations. In order to ensure the validity of the semiclassical or one loop approximation, it will be necessary that the off-diagonal fluctuations have relatively large masses. From the Lagrangian (2.9), we can estimate the masses of the light off-diagonal excitations of the zero momentum scalar modes to be

$$m_{ij}^2 \sim \left(\varphi_{1,ij}^2 + \varphi_{2,ij}^2\right) + R^{-2} - \mu_1^2 \tag{3.8}$$

where $\varphi_{a,ij} \equiv \varphi_{ai} - \varphi_{aj}$. Validity of perturbation theory in the Wilsonian sense, about diagonal scalar backgrounds then requires that the off diagonal modes be much heavier than the light diagonal degrees of freedom:

$$R^2 \sum_p \left(\varphi_{1,ij}^2 + \varphi_{2,ij}^2\right) \gg |1 - (\mu_1 R)^2| \simeq \mathcal{O}(\lambda). \tag{3.9}$$

This condition is easy to ensure in the near critical region (3.5) by choosing appropriately large VEVs for all the diagonal entries, such that the differences between them also remain parametrically large.

The masses of all heavy excitations will depend only on the sum

$$\varphi_{ij}^2 \equiv \left(\varphi_{1,ij}^2 + \varphi_{2,ij}^2\right), \tag{3.10}$$

as seen above.[4] For this reason, at leading order in the coupling, the one-loop contribution to the effective potential for light modes only depends on this combination. In this case, we can use the resulting symmetry to only turn on a VEV for one of the scalar fields, which we can take to be φ_1, for instance. The full dependence on the scalar fields can then be reconstructed by replacing φ_{ij}^2 (we drop the subscript 1) by (3.10).

The radiative corrections at the one loop level are then obtained by taking the constant modes as backgound VEVs and integrating out all the massive modes of the fields. To this end, we shift $\phi_1 \to \phi_1 + \varphi$ and at non-zero temperature, we also introduce a zero mode for the time component of the gauge field via $A_0 \to A_0 + \alpha$. The VEVs contribute to the effective masses for the modes and the one-loop correction involves the logarithm of the resulting fluctuation determinants and hence depends on the VEVs in a non-trivial way. It is important to note to leading order in the coupling in the near critical region, we can take $\mu_1 = R^{-1}$ exactly.

Before we compute these determinants, we must first fix the gauge. We find it convenient to do this by working in a conventional R_ξ gauge of a spontaneously broken gauge theory and then to specialize to Feynman gauge $\xi = 1$.[5] We add to the action the gauge fixing term

$$\mathcal{L}^{(\mathrm{gf})} = \frac{1}{2g^2}\mathrm{Tr}\Big(\nabla_i A^i + \tilde{D}_0 A^0 - i\varphi\phi_1\Big)^2 \,. \tag{3.11}$$

In the above equation and in what follows, we leave adjoint action by φ as implicit, i.e. $\varphi\phi \equiv [\varphi, \phi]$, $\varphi^2\phi \equiv [\varphi, [\varphi, \phi]]$, etc. In addition, $\tilde{D}_0 = \partial_0 + i\alpha$ includes α the zero mode part of A_0 only, and, as for φ, adjoint action for α is implied.

Although in the absence of chemical potentials the gauge fixing removes cross terms between the gauge field and the scalars, the presence of a chemical potential and a VEV φ introduces additional ones that are not removed by the gauge fixing term. The modes A_0, ϕ_1 and ϕ_2 are all coupled together as seen from the expression for the bosonic part of the action at Gaussian order in quantum fluctuations:

$$\begin{aligned}\mathcal{L}^{(\mathrm{boson})} = \frac{1}{g^2}\mathrm{Tr}\,\Big[&\frac{1}{2}A_0(-\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2)A_0 + \frac{1}{2}A_i(-\tilde{D}_0^2 - \Delta^{(v)} + \varphi^2)A_i \\ &+ \frac{1}{2}\phi_1(-\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2)\phi_1 + \frac{1}{2}\phi_2(-\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2)\phi_2 \\ &+ R^{-1}(A_0\varphi\phi_2 - \phi_2\varphi A_0 + i\phi_1\tilde{D}_0\phi_2 - i\phi_2\tilde{D}_0\phi_1) \\ &+ \frac{1}{2}\sum_{a=3}^{6}\phi_a(-\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2 + R^{-2})\phi_a + \bar{c}(-\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2)c\Big].\end{aligned} \tag{3.12}$$

Here we have set $\mu_1 = R^{-1}$, $\Delta^{(s)}$ and $\Delta^{(v)}$ are the scalar and vector Laplacians on S^3, and we have explicitly included the ghosts $c, \bar{c}$ in the Lagrangean. The detailed properties of

[4] In the presence of diagonal background VEVs for the lightest scalars, *all* the degrees of freedom which are integrated out are necessarily the off-diagonal fluctuations of every KK harmonic on S^3.

[5] We shall not be unduly concerned by the usual ξ dependence of the effective potential that generally plagues gauge theories. The reason is that in the vicinity of the critical region there is an exactly flat direction at tree level and so the VEV can be taken to be non-zero whilst remaining on shell. The ξ dependence then drops out as one can explicitly find by including it in all subsequent steps.

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these Laplacians and their eigenvalues are summarized in appendix A. We need only to note that while $\Delta^{(s)}$ has an $\ell = 0$ mode, the vector Laplacian $\Delta^{(v)}$ on S^3 does not have a zero mode. Furthermore, the vector fluctuations A_i can be decomposed into the image and kernel of the gradient operator ∇^i as $A^i = B^i + C^i$ with with $\nabla_i B^i = 0$ and $C^i = \nabla^i f$.

The fermionic fluctuations about the non-zero backgrounds for φ_1 and φ_2 are governed by the Lagrangean,

$$\mathcal{L}^{(\text{fermion})} = \text{Tr}\left(i\bar{\psi}_A\left(\not{D} - \bar{\mu}_A\gamma_0\gamma_5\right)\psi_A - \bar{\psi}_A\left(\alpha^1_{AB}\varphi_1 + i\beta^1_{AB}\gamma_5\varphi_2\right)\psi_B\right), \tag{3.13}$$

where

$$\bar{\mu}_A = R^{-1}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right), \tag{3.14}$$

using (2.8). The ψ^A are four component Majorana fermions.

The one loop contribution to the effective potential is obtained by integrating out each *massive* fluctuation, giving rise to the corresponding fluctuation determinant. Importantly, the quadratic fluctuation operators for the (A_0, ϕ_1, ϕ_2) sector and the fermionic sector are off-diagonal in flavour space. For example, using that the eigenvalues of $\Delta^{(s)}$ are given by $\ell(\ell+2)R^{-2}$, for $\ell = 0, 1, \ldots$, we can evaluate the determinant of the fluctuation operator coupling the (A_0, ϕ_1, ϕ_2) sector for a given spherical harmonic number,

$$\begin{aligned}\det &\begin{pmatrix} -\tilde{D}_0^2 + \ell(\ell+2)R^{-2} + \varphi^2 & 0 & 2R^{-1}\varphi \\ 0 & -\tilde{D}_0^2 + \ell(\ell+2)R^{-2} + \varphi^2 & 2iR^{-1}\tilde{D}_0 \\ -2R^{-1}\varphi & -2iR^{-1}\tilde{D}_0 & -\tilde{D}_0^2 + \ell(\ell+2)R^{-2} + \varphi^2 \end{pmatrix} \\ &= \left[(-\tilde{D}_0^2 + \ell(\ell+2)R^{-2} + \varphi^2)(-\tilde{D}_0^2 + \ell^2 R^{-2} + \varphi^2)(-\tilde{D}_0^2 + (\ell+2)^2 R^{-2} + \varphi^2)\right].\end{aligned} \tag{3.15}$$

The zeroes of this expression viewed as a polynomial in $\tilde{D}_0$ yield precisely the energies of the harmonics. Note that for this, the simplest situation at hand, these match the results summarized in table 1, if we set $\mu_1 = R^{-1}, \mu_2 = \mu_3 = 0$. The table shows that introducing a chemical potential leads to a re-organization of the energy levels. In the bosonic sector, when $\mu_1 = R^{-1}$ is turned on, the result is three new towers of modes that are shifted in such a way that one of the new towers is identical to the original A_0 tower (which can be cancelled with the C_i and ghosts) while the other two new towers are the deformations of the original ϕ_1 and ϕ_2 towers. For fermions, notice that the original half-integer graded modes are now integer graded.

The fluctuation determinant for each species then enters the effective potential at one loop as

$$V_1 = \frac{T}{2\pi^2 R^3}\frac{1}{2}\sum_{\text{species}}\sum_{ij=1}^{N}\sum_{\ell=\ell_0}^{\infty} d_\ell^{B(F)}\,\log\det\left[-\tilde{D}_0^2 + \varepsilon_\ell(\varphi_{ij})^2\right], \tag{3.16}$$

where $d_\ell^{B(F)}$ is the degeneracy of bosonic (fermionic) modes with angular momentum quantum number ℓ. The integer ℓ_0 is the lower limit on the angular momentum quantum number. The quantity ε_ℓ is the energy of the mode in question. The degeneracy factors $d_\ell^{B(F)}$ are positive or negative depending upon the the statistics of the corresponding fields.

Field	d_ℓ	ε_ℓ	ℓ_0
B_i	$2\ell(\ell+2)$	$\sqrt{R^{-2}(\ell+1)^2+\varphi^2}$	1
C_i	$(\ell+1)^2$	$\sqrt{R^{-2}\ell(\ell+2)+\varphi^2}$	1
$(c,\bar{c})$	$-2(\ell+1)^2$	$\sqrt{R^{-2}\ell(\ell+2)+\varphi^2}$	0
$(A_0,\phi_{1,2})_1$	$(\ell+1)^2$	$\sqrt{R^{-2}\ell(\ell+2)+\varphi^2}$	0
$(A_0,\phi_{1,2})_{2,3}$	$(\ell+1)^2$	$\sqrt{R^{-2}(\ell+1\pm R\mu_1)^2+\varphi^2}$	0
$\phi_{3,4}$	$(\ell+1)^2$	$\sqrt{R^{-2}(\ell+1)^2+\varphi^2}\pm\mu_2$	0
$\phi_{5,6}$	$(\ell+1)^2$	$\sqrt{R^{-2}(\ell+1)^2+\varphi^2}\pm\mu_3$	0
ψ^A_α	$-\ell(\ell+1)$	$\sqrt{R^{-2}(\ell+\frac{1}{2}\pm\frac{1}{2}R\mu_1)^2+\varphi^2}\pm\frac{\mu_2}{2}\pm\frac{\mu_3}{2}$	1

Table 1: The fields, their degeneracy and energies as a function of the chemical potentials with a nonvanishing VEV for ϕ_1. The expressions for the mode energies are only valid when either $\mu_p = 0$ or $\mu_p = R^{-1}$.

In table 1 we summarize the data associated to each set of modes. We emphasize that these energies are strictly only correct when either $\mu_p = 0$ or $\mu_p = R^{-1}$. Where sign choices exist all the possible combinations must be taken.

The contributions to the effective action are standard expressions in thermal field theory. First of all, the eigenvalues of $i\partial_0$ are $2\pi n/\beta$, for $n \in \mathbb{Z}$ for bosons. When acting on fermions, due to antiperiodic boundary conditions for fermions around the thermal S^1, the operator $i\partial_0$ has eigenvalues $2\pi(n+1/2)/\beta$, for $n \in \mathbb{Z}$. It is standard practice in thermal field theory to perform a Poisson resummation over n in such a way that each contribution splits into a piece that describes the theory at $T = 0$ and the non-trivial "thermal" part which vanishes as $TR \to 0$. The zero temperatue piece is the Casimir energy in the presence of background expectation values for fields. So a typical term in the one loop potential (3.16) can be expressed as

$$\frac{1}{\mathrm{Vol}(S^3)}\sum_{\ell=\ell_0}^{\infty} d_\ell \log\,\det(-\tilde{D}_0^2+\varepsilon_\ell(\varphi)^2) = \tag{3.17}$$

$$\text{Bosonic}: \quad \frac{1}{\mathrm{Vol}(S^3)}\frac{1}{2}\sum_{ij=1}^{N}\sum_{\ell=\ell_0}^{\infty} d_\ell^B\left\{|\varepsilon_\ell(\varphi_{ij})| - \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\beta|\varepsilon_\ell(\varphi_{ij})|}\cos(n\alpha_{ij}\beta)\right\}\,,$$

$$\text{Fermionic}: \quad \frac{1}{\mathrm{Vol}(S^3)}\frac{1}{2}\sum_{ij=1}^{N}\sum_{\ell=\ell_0}^{\infty} d_\ell^F\left\{|\varepsilon_\ell(\varphi_{ij})| - \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n\beta|\varepsilon_\ell(\varphi_{ij})|}\cos(n\alpha_{ij}\beta)\right\}$$

where we have written the adjoint trace explicitly and defined $\varphi_{ij}\equiv\varphi_i-\varphi_j$ and $\alpha_{ij}=\alpha_i-\alpha_j$.

More than one non-zero critical chemical potential

The analysis with more than one non-zero critical chemical potential proceeds similarly to the above. The only difference lies in the mixing matrices for bosonic and fermionic fluctuations and their resulting eigenvalues.

When $\mu_1 \simeq \mu_2 \simeq R^{-1}$ and $\mu_3 = 0$, the light scalar modes are $(\varphi_1, \ldots \varphi_4)$ and a 5×5 mixing matrix results for the fluctuations in the bosonic sector. The zeroes of the determi-

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nant of this mixing matrix in frequency space correspond to the mode energies. Similarly one may find the fluctuation energies for the case of three critical chemical potentials when all six scalar fluctuations and A_0 are coupled. The results for all the mode energies, both bosonic and fermionic are listed in table 1.

4. The zero temperature effective potential

As we take the low temperature limit of (3.17), the thermal contributions are exponentially suppressed by the Boltzmann factors. Then we are only left with a Casimir energy sum that yields the one loop effective potential

$$V_1(T=0) = \frac{1}{\mathrm{Vol}(S^3)}\frac{1}{2}\sum_{\text{Species}}\sum_{ij=1}^{N}\sum_{\ell=\ell_0}^{\infty} d_\ell^{B(F)}|\varepsilon_\ell(\varphi_{ij})|. \tag{4.1}$$

As is well known, this sum is formally divergent and needs to be regulated.

The choice of regulator is a subtle issue especially in the presence of chemical potentials and there is more than one way to regulate such sums. We choose to cut off the sums keeping two crucial points in mind: First, the cutoff will be imposed on the *energies* of the modes rather than the angular momenta ℓ so that the regulator is general coordinate invariant.[6] Secondly, the energy cutoff function will be chosen to be completely independent of the chemical potential(s). What this means is that, all mode sums will be regulated using the cutoff functions of the theory with $\mu_1 = \mu_2 = \mu_3 = 0$. This requirement may be motivated by the physical observation that introducing a chemical potential is a deformation of the *state* of the theory and not the functional integral measure.

For each field type, we introduce the regulated expression for the energy sums

$$\mathscr{E}_{\text{reg}} = \frac{1}{2}\sum_{ij=1}^{N}\sum_{\ell=\ell_0}^{\infty} d_\ell^{B(F)}\,|\varepsilon_\ell(\varphi_{ij})| f(\varepsilon_\ell^{(0)}/\Lambda). \tag{4.2}$$

Here $f(x)$ can be thought of as the smooth version of a cutoff function which is unity for $x < 1$ and 0 for $x > 1$ such that $f(0) = 1$ while all the derivatives $f'(0) = f''(0) = \cdots = 0$. Crucially, the cutoff is on the energy $\varepsilon_\ell^{(0)}$ which is defined to be the energy of the respective mode with $\mu_p = 0$.

We can evaluate the regulated sum by using the Abel-Plana formula [28] appropriate to a function with branch cuts on the imaginary axis or the left half plane

$$\sum_{n=0}^{\infty} F(n) = \int_0^\infty dx\, F(x) + \frac{1}{2}F(0) - 2\int_0^\infty dx\, \frac{\mathrm{Im}\, F(ix)}{e^{2\pi x}-1}\ . \tag{4.3}$$

Let us now turn to the evaluation of each field contribution to the Casimir energy seriatim.

The first point that is immediately clear is that large cancellations occur between the C_i, the ghosts $(c, \bar{c})$ and $(A_0, \phi_1, \phi_2)_1$ which is basically the A_0 field. Indeed, all the $\ell > 0$ contributions cancel between these, leaving a net result $-\frac{1}{2}|\varphi|$.

[6] See the footnote (30) in [6] and section 6 of [27] for a discussion of these issues.

The regularized contributions from the B_i and the scalar fields remain in the bosonic sector. Taking care to use the $\mu_p = 0$ energies in the cutoff functions and applying the Abel-Plana formula, we find that all bosonic modes together induce the one loop potential

$$\begin{aligned}\mathscr{E}_B =&\Lambda^4 R^3 - \frac{1}{2}R\Lambda^2 - R^3\varphi^2\Lambda^2 + \frac{1}{12R} - \frac{1}{4}\varphi^2 R \\ &+ \frac{1}{2}\varphi^4 R^3 \log\left(\frac{|\varphi|e^{1/4}}{2\Lambda}\right) + 8\int_{R\varphi}^{\infty} \frac{x^2\sqrt{x^2R^{-2}-\varphi^2}}{e^{2\pi x}-1}.\end{aligned} \tag{4.4}$$

Interestingly, the potentially problematic linear term $-\frac{1}{2}|\varphi|$ from the ghosts, A_0 and the C_i, cancels off in the full sum. The result is valid for any number of critical chemical potentials. The sum over energies for more than one critical chemical potential reduces to the one for a single chemical potential due to a simple cancellation of the dependence on μ_2 and μ_3. This cancellation is obvious from the fluctuation energies of $\phi_3, \ldots \phi_6$ listed in table 1.

For the fermions, again employing a cutoff on the mode energies at zero chemical potential, the Casimir energy is

$$\begin{aligned}\mathscr{E}_F = &-\Lambda^4 R^3 + \frac{1}{2}R\Lambda^2 + R^3\varphi^2\Lambda^2 + \frac{5}{48R} + \frac{1}{4}\varphi^2 R \\ &- \frac{1}{2}\varphi^4 R^3 \log\left(\frac{|\varphi|e^{1/4}}{2\Lambda}\right) - 8\int_{R\varphi}^{\infty} \frac{x^2\sqrt{x^2R^{-2}-\varphi^2}}{e^{2\pi x}-1}.\end{aligned} \tag{4.5}$$

The total one loop quantum effective potential on S^3 is therefore

$$\frac{1}{\mathrm{Vol}(S^3)}\left(\mathscr{E}_F + \mathscr{E}_B\right) = \frac{1}{\mathrm{Vol}(S^3)}\,\frac{3}{16R}. \tag{4.6}$$

This is remarkable since all the dependence on the background VEVs has completely cancelled out to yield the zero point energy of $\mathcal{N} = 4$ theory on S^3. This implies that at least at one loop order, at the critical chemical potential, there is a complete cancellation between bosonic and fermionic quantum fluctuations resulting in flat directions in the quantum effective potential.[7]

To summarize the result of the computation presented in this section: We have found that at the critical value for one, two or all three chemical potentials of $\mathcal{N} = 4$ theory on S^3 at *zero temperature*, there is no radiatively induced potential for the scalar fields at one loop order. This implies, at least at the one loop order, a Coulomb branch "moduli space" of complex dimension N, $2N$ or $3N$, depending on whether one, two or three chemical potentials are at the critical value.

Below we offer a natural explanation for the appearance of these flat directions in the effective potential calculation, and its existence to all orders in the interacting theory at zero temperature.

[7]It is worth contrasting this result with the same calculation done at zero temperature and vanishing chemical potentials. In that case one obtains [13] a complicated one loop effective potential on S^3.

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4.1 Massless BPS states at critical chemical potential

At zero temperature, chemical potentials for the three $\mathrm{U}(1)_R$ charges deform the Hamiltonian on S^3 as,

$$\Delta \to \Delta - \sum_{p=1}^{3} \mu_p J_p \,. \tag{4.7}$$

The $\mathcal{N} = 4$ superconformal algebra [22] includes the following anticommutation relation between the superconformal generators $S_{A\alpha}$ and the supercharges $Q^{B\beta}$. Noting that on S^3, following the rules of radial quantization, $\left(Q^{A\alpha}\right)^\dagger = S_{A\alpha}$, we have

$$\{Q^{A\alpha\dagger}, Q^{B\beta}\} = \delta_A^B\ \delta_\alpha^\beta\ \Delta + \sum_p \left(J_p\, R_p\right)_A^B\ \delta_\alpha^\beta + \ \delta_A^B\ \left(\mathcal{J}_1\right)_\alpha^\beta \,. \tag{4.8}$$

A similar relation holds for $\bar{Q}$ and $\bar{S}$. The indices $A, B = 1 \ldots 4$ are $\mathrm{SU}(4)_R$ indices and R_p $(p = 1, 2, 3)$ are the three $\mathrm{U}(1)_R$ Cartan generators in the fundamental representation of the $\mathrm{SU}(4)_R$ algebra; $\mathcal{J}_1$ is a generator of an $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ isometry group of S^3.

For a subset of the supercharges, the right hand side of the above relation vanishes on BPS states of the $\mathcal{N} = 4$ theory. Multi-trace operators with R charge $(J_1, 0, 0)$ and with $(R\Delta) - J_1 = 0$ are $\frac{1}{2}$ BPS states which are annihilated by one half of the chiral supercharges in (4.8). Similarly, states satisfying $(R\Delta) - J_1 - J_2 = 0$ and $(R\Delta) - J_1 - J_2 - J_3 = 0$ correspond to $\frac{1}{4}^{\text{th}}$ BPS and $\frac{1}{8}^{\text{th}}$ BPS states respectively.

The operators $\Delta - J_1/R$; $\Delta - (J_1 + J_2)/R$ and $\Delta - (J_1 + J_2 + J_3)/R$ are of course the Hamiltonians of the theory with critical chemical potentials (4.7). It is thus clear that the ground states of the Hamiltonian with critical chemical potentials are infinite sets of one half, one quarter and one-eighth BPS states of the $\mathcal{N} = 4$ theory, depending on whether we have one, two or three critical chemical potentials. Importantly, in each case there is an infinite degeneracy of ground states, all of which have zero energy, following from (4.8).

Choosing an appropriate $\mathcal{N} = 1$ subalgebra of the $\mathcal{N} = 4$ superalgebra, the chiral ring is the set of holomorphic gauge invariant operators made up of polynomials of $\Phi_1 = \phi_1 + i\phi_2$, $\Phi_2 = \phi_3 + i\phi_4$ and $\Phi_3 = \phi_5 + i\phi_6$, and the chiral gauge field strength W_α, modulo F-term relations (on $\mathbb{R}^4$). These are the $\frac{1}{8}$ BPS states. On $\mathbb{R}^4$, the chiral primary operators at a generic point on the Coulomb branch of the theory, are the the same as the operators of the chiral ring described above. The F-term conditions ensure that the fields may be simultaneously diagonalized and the chiral operators are distinct polynomials of the $3N$ bosonic and $2N$ fermionic diagonal eigenvalues, invariant under the permutation group S_N. On S^3, these operators are built out of the s-waves of the diagonal elements of the three complex scalars and two polarizations of the $\ell = 1$ harmonic of a fermion [29].

Similarly, the half BPS states correspond to S_N invariant polynomials involving the diagonal elements of the s-wave of a single bosonic holomorphic field, namely $\varphi_1 + i\varphi_2$. The quarter BPS states are generated by diagonal elements of s-waves of the holomorphic bosonic operators $\varphi_1 + i\varphi_2$ and $\varphi_3 + i\varphi_4$.

The dimensions of the moduli spaces encountered at the different numbers of critical chemical potentials, and the corresponding counting of zero modes is therefore perfectly consistent with the interpretation that at critical chemical potential BPS states of the

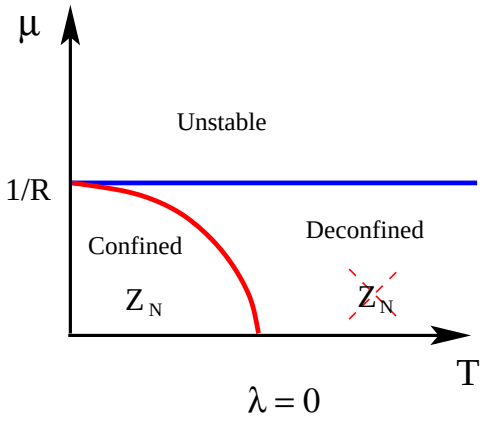

Figure 1: The $\lambda = 0$ phase diagram as a function of temperature and chemical potential, found by [11].

$\mathcal{N} = 4$ theory become light. Each point on the resulting moduli space represents a coherent superposition of the light BPS states.

5. Low temperatures and a metastable plasma phase

To summarize the story thus far, the phase diagram of the free theory was studied in [11] at all temperatures and chemical potentials. At infinite N, the authors of [11] established the existence of a first order Hagedorn/deconfinement transition for any fixed μ_p as the temperature is increased (see figure 1.). For $\mu_p > R^{-1}$, the free theory ceases to have a well-defined ground state in the grand canonical ensemble.

In the vicinity of $\mu_p \simeq R^{-1}$, we have shown that light scalars appear and remain light in the interacting theory at $T = 0$. Furthermore, we see flat directions associated to these light modes. For a small non-zero temperature $TR \ll 1$, we expect the scalar modes $(\varphi_{2p-1}, \varphi_{2p})$ to continue to be light degrees of freedom in the near critical region. In addition there will be another set of zero modes in the theory, namely the α_i, the diagonal elements of the Polyakov loop matrix.

5.1 Low temperatures and $\mu_1 = R^{-1}, \mu_2 = \mu_3 = 0$

At the critical chemical potential, and at zero temperature, the effective potential is flat including quantum effects. There are no non-analyticities in the effective potential near the origin, even when off-diagonal modes become light. This is due to an exact cancellation between the Vandermonde measure factor (equivalently the ghosts and A_0) and the contribution from the new scalar zero mode which appears at the critical chemical potential.

Our strategy will be to stay at $\mu_1 = R^{-1}, \mu_2 = \mu_3 = 0$ and switch on a small non zero

temperature $TR \ll 1$. The finite temperature effective potential at this critical point is

$$V_0 + V_1 = \sum_{ij=1}^{N} \frac{1}{\mathrm{Vol}(S^3)} \left[\frac{3}{16R} - 8T \sum_{k=1}^{\infty} \frac{1}{2k-1} \cos\left((2k-1)\frac{\alpha_{ij}}{T}\right) \sum_{\ell=1}^{\infty} \ell^2 e^{-\frac{(2k-1)}{T}\sqrt{\ell^2 R^{-2} + \varphi_{ij}^2}} \right]. \quad (5.1)$$

The effective potential above is a function of the light fields

$$\varphi_{ij}^2 = (\varphi_{1i} - \varphi_{1j})^2 + (\varphi_{2i} - \varphi_{2j})^2. \quad (5.2)$$

A notable feature of this expression is that there are no non-analytic terms near the origin where we expect off-diagonal modes of ϕ_1 and ϕ_2 to become light. The exact cancellation between the scalar zero mode and measure factors persists at non-zero temperature. This is related to the fact that with one critical chemical potential, the ground states at zero temperature are parametrized by multi-trace gauge-invariant combinations of the s-wave of a *single* holomorphic field, namely $\varphi_1 + i\varphi_2$. All other modes are heavy. At the origin, therefore, we expect that even if off-diagonal modes of the field appear to become light at non-zero temperature, we may gauge rotate them away completely. (By the same logic, we expect not to be able to do this for the $\frac{1}{4}^{\text{th}}$ and $\frac{1}{8}^{\text{th}}$ BPS states, since they involve multi-trace combinations of more than one holomorphic field.)

The radiative corrections vanish exponentially at large field amplitudes $|\varphi_{ij}|R \gg 1$. This is actually a generic, robust feature expected of the effective potential of $\mathcal{N} = 4$ theory on S^3. In the limit of large field amplitudes $|\varphi_{ij}|R \gg 1$, all off-diagonal modes are proportionately heavier and therefore should decouple from the theory. That this happens unambiguously is linked to the UV finiteness of the $\mathcal{N} = 4$ theory. Furthermore, as already noted above, the putative scalar zero energy mode (near $\varphi_{ij}^2 \approx 0$) has cancelled against the ghosts.

Now we need to determine the minimum of this joint potential. It is clear that the α_i experience a purely attractive pairwise potential near $\varphi_{ij} = 0$. This is because the repulsive Vandermonde determinants have been eliminated through cancellations with the zero mode. Hence at finite temperature, at the critical chemical potential,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_N; \quad (5.3)$$

which means that the large N theory should be thought of as being deconfined and the large N distribution of the α_i is a delta function. With $\alpha_{ij} = 0$, it is clear that at $\varphi_{ij} = 0$, the scalars have a mass squared which is strictly positive and the origin is the stable vacuum of the theory.

Expanding the potential for $R^2\varphi_{ij}^2 \ll 1$, and $TR \ll 1$, we see that thermal effects contribute a small mass to the scalars

$$V_0 + V_1 \approx \frac{N^2}{2\pi^2 R^4} \left(\frac{3}{16} - 8TR\, e^{-\frac{1}{TR}} + 2\, R^2\, e^{-\frac{1}{TR}} \frac{1}{N} \sum_{i=1}^{N} (\varphi_{1i}^2 + \varphi_{2i}^2) \right). \quad (5.4)$$

Interestingly, a small temperature $TR \ll 1$ has lowered the vacuum energy at the origin to $3/16R - 8Te^{-1/TR}$ and the light scalars φ_1 and φ_2 have acquired an exponentially small positive mass.

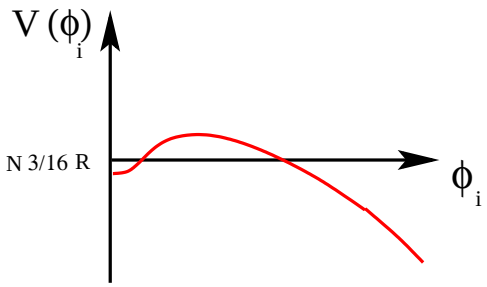

Figure 2: The low temperature $TR \ll 1$ effective potential for one diagonal mode of the light scalar with $0 < \mu_1 - R^{-1} \leq (4\lambda/\pi^2 R)\exp(-\frac{1}{TR})$ and $\mu_2 = \mu_3 = 0$.

Metastable plasma for $\mu_1 \gtrsim R^{-1}, \mu_2 = \mu_3 = 0$

We now want to argue that since a small temperature makes the light scalars massive at $\mu_1 = R^{-1}$, the origin will continue to be a locally stable vacuum if we raise μ_1 by a sufficiently small amount above its critical value. This is of course only possible if the thermal mass can beat the instability induced by the chemical potential. Crucially, we also require that, for our analysis to apply

$$\mu_1 - R^{-1} \lesssim \mathcal{O}(\lambda), \tag{5.5}$$

which allows us to set $\mu_1 = R^{-1}$ in the one loop calculation, the error in doing so appearing at one higher order in perturbation theory.

With this choice of parameters, we have

$$V_0 + V_1 = \sum_{ij=1}^{N} \left(\frac{1}{4\lambda}(R^{-2} - \mu_1^2)\varphi_{ij}^2 + \frac{1}{\mathrm{Vol}(S^3)} \left[\frac{3}{16R} - 8T\, e^{-\frac{1}{TR}\sqrt{1+R^2\varphi_{ij}^2}} \right] \right). \tag{5.6}$$

Since the quantum correction vanishes exponentially at large field amplitudes, the potential will revert to its classical runaway behaviour in these regions whenever $\mu_1 > R^{-1}$. Close to the origin, however, the situation is different. For the following range of values of μ_1

$$0 < \mu_1 - R^{-1} < \frac{4\lambda}{\pi^2 R}\, e^{-\frac{1}{TR}}, \tag{5.7}$$

the theory has a metastable vacuum at the origin. The behaviour of the effective potential is shown in figure 2.

It is reasonable to assume that this metastable phase is the continuation to low temperatures, of the high temperature metastability found in [11]. As in the high temperature situation, the lifetime of the state is determined by the probability for one diagonal mode, say φ_{1i}, to make its way out into the unstable region. This can happen either by thermal activation over the barrier, or by tunnelling through the barrier. The probability for thermal activation can be estimated by computing the Boltzmann suppression factor. At low temperatures , the height of the barrier is exponentially suppressed and $\propto \exp(-\frac{1}{TR})$, so that the associated probability for thermally exciting one eigenvalue goes as $\exp\left(-Ne^{-1/TR}\right)$.

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Estimating the barrier penetration probability is more complicated due to the form of the effective potential, but the dependence on N is obvious, due to the scaling of the action with N. The lifetime of the metastable plasma diverges in the strict limit $N \to \infty$, as was also found for the high temperature metastable phase in [11].

5.2 Two and three non-zero critical chemical potentials

We have seen that with two and three critical μ_p, the theory at $T = 0$ has flat directions which are not lifted by quantum corrections.

At finite temperature however, the one-loop effective potential for the diagonal modes with two and three critical chemical potentials displays some qualitatively different features compared to the case with one chemical potential. The main difference is due to the appearance of additional zero modes for small φ_{ij}. The potential for the diagonal modes exhibits non-analytic behaviour near the origin. We may understand these non-analyticities as being due to the off-diagonal modes of the s-wave components of the light holomorphic fields getting excited near the origin, at finite temperature.

At zero temperature and critcial μ_p, the light Coulomb branch states consist of $\frac{1}{4}^{\text{th}}$ and $\frac{1}{8}^{\text{th}}$ BPS states. However, close to $\varphi_{ij} = 0$ at finite temperature, there are also light off-diagonal excitations which are non-BPS. Near the origin, therefore, it is more appropriate to look at the perturbative result for the effective potential for the s-wave of the full matrix fields obtained by integrating out higher harmonics. As we will see below the only effect of this is to provide a small positive thermal mass.

Two critical chemical potentials. The low temperature analysis ($TR \ll 1$) for two near critical chemical potentials yields the following one loop correction to the effective potential for the diagonal modes,

$$V_1 = \sum_{ij=1}^{N} \frac{1}{\text{Vol}(S^3)} \left[\frac{3}{16R} - \frac{T}{2} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{n\alpha_{ij}}{T}\right) e^{-\frac{n}{TR}\sqrt{1+\varphi_{ij}^2 R^2}+\frac{n}{TR}} \right. \\ \left. - 2\,T e^{-\frac{1}{TR}\sqrt{1+\varphi_{ij}^2 R^2}+\frac{1}{2TR}} \cos\left(\frac{\alpha_{ij}}{T}\right) + \mathcal{O}(e^{-\frac{1}{TR}}) \right]. \tag{5.8}$$

Here $\varphi_{ij}^2 = \sum_{a=1}^{4} (\varphi_{ai} - \varphi_{aj})^2$. Once again, near $\varphi_{ij} = 0$, the α_i experience a purely attractive potential so that $\alpha_i = \alpha_j$ and the theory is in the deconfined phase. This expression for the effective potential exhibits non-analytic behaviour at the origin. With $\alpha_{ij} = 0$, we have

$$V_1 = \sum_{ij=1}^{N} \frac{1}{\text{Vol}(S^3)} \left[\frac{3}{16R} + \frac{T}{2} \log\left(1 - e^{-\frac{1}{TR}\sqrt{1+\varphi_{ij}^2 R^2}+\frac{1}{TR}}\right) \right. \\ \left. - 2T\, e^{-\frac{1}{TR}\sqrt{1+\varphi_{ij}^2 R^2}+\frac{1}{2TR}} + \cdots \right]. \tag{5.9}$$

Near the origin, the logarithm provides an infinite *attractive* potential and is the result of integrating out off-diagonal elements of the $\ell = 0$ harmonic of $\varphi_3 + i\varphi_4$. The second term

which is $\mathcal{O}(e^{-1/2RT})$, originates from integrating out the next lightest state, namely the fermion with mass $1/2R$. Expanding this contribution about $\phi_{ij} = 0$,

$$-\frac{1}{\mathrm{Vol}(S^3)}\, 2\, T e^{-\frac{1}{TR}\sqrt{1+\varphi_{ij}^2 R^2}+\frac{1}{2RT}} \approx \frac{1}{2\pi^2 R^3}\left(-2T\, e^{-\frac{1}{2RT}} + R\, e^{-\frac{1}{2RT}}\, \varphi_{ij}^2\right), \tag{5.10}$$

we see a thermal mass for the diagonal modes. By gauge invariance at the origin, the off-diagonal fluctuations will also obtain the same thermal mass. The resulting picture is therefore rather similar to what we have seen earlier. Instead of integrating out light off-diagonal modes near the origin, we may simply compute the thermal mass of the lightest fields, namely $\varphi_1 + i\varphi_2$ and $\varphi_3 + i\varphi_4$ by computing the usual Feynman one-loop self-energy graphs at finite temperature. At quadratic order, the thermal effective potential for these fields close to the origin will have the form

$$V = \frac{N^2}{2\pi^2 R^3}\left(\frac{3}{16R} - 2T\, e^{-\frac{1}{2RT}} + 2\, R\, e^{-\frac{1}{2RT}}\, \sum_{p=1}^{4} \frac{1}{N}\mathrm{Tr}(\varphi_a^2)\; +\; \text{quartic}\right). \tag{5.11}$$

The higher order interactions are the terms responsible for inducing non-analyticities in the effective potential for the diagonal modes when off-diagonal fluctuations are (wrongly) integrated out close to the origin of field space.

Hence, a finite temperature introduces a small dip in the potential energy at the origin, which then asymptotes to the constant value of $3/(16R)$ at large field amplitudes in accordance with (5.8).

Three critical chemical potentials. The situation with three critical chemical potentials is similar to that with two critical chemical potentials. There is an additional light scalar as well as a fermion zero mode. The potential for the light diagonal modes is

$$\begin{aligned} V_1 = \sum_{ij=1}^{N}\Bigg(\frac{1}{\mathrm{Vol}(S^3)}\Bigg[\frac{3}{16R} - 2\,T\sum_{k=1}^{\infty}\frac{1}{2k-1}\cos\left((2k-1)\frac{\alpha_{ij}}{T}\right) e^{-\frac{(2k-1)}{RT}\left[\sqrt{1+\varphi_{ij}^2R^2}-1\right]} \\ - 4T\left(e^{-\frac{1}{TR}\sqrt{1+\varphi_{ij}^2R^2}} + 2e^{-\frac{1}{TR}\left[\sqrt{1+\varphi_{ij}^2R^2}-1\right]}\right)\cos\left(\frac{\alpha_{ij}}{T}\right) + \cdots\Bigg]\Bigg). \end{aligned} \tag{5.12}$$

As before the α_i experience an attractive potential resulting in a delta-function distribution at large N for these fields. There is then a logarithmic attractive potential between the light scalar diagonal modes. It is therefore more appropriate to keep the all s-wave fluctuations of the matrices $\varphi_1, \varphi_2, \ldots \varphi_6$ while integrating out all higher partial waves on S^3. This leads to a temperature induced positive curvature in the effective potential near the origin

$$V \approx \frac{N^2}{2\pi^2 R^3}\left(\frac{3}{16R} - 12\, T\, e^{-\frac{1}{TR}} + 8\, e^{-\frac{1}{TR}}\sum_{a=1}^{6}\frac{1}{N}\mathrm{Tr}\varphi_a^2 + \cdots\right). \tag{5.13}$$

At large field amplitudes the effective potential approaches a constant, given by the Casimir energy of the $\mathcal{N} = 4$ theory on the sphere.

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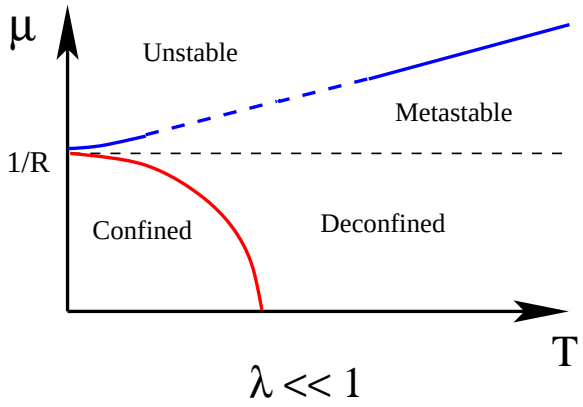

Figure 3: The known features of the weak coupling phase diagram of $\mathcal{N} = 4$ theory with a chemical potential.

Metastable phase

Since we have established a positive curvature for the effective potential at the origin, the argument for the existence of a metastable plasma phase above critical chemical potential proceeds exactly as in the situation with one chemical potential.

For two equal chemical potentials, the metastable phase exists provided the larger of the two chemical potentials exists in the range

$$0 < \mu - R^{-1} < \frac{2\lambda}{\pi^2 R}\, e^{-\frac{1}{2TR}}. \tag{5.14}$$

For three chemical potentials, the metastable plasma phase exists if the largest of the chemical potentials satisfies

$$0 < \mu - R^{-1} < \frac{8\lambda}{\pi^2 R}\, e^{-\frac{1}{TR}}. \tag{5.15}$$

In both cases, the lifetime of this phase is determined by the probability for one diagonal mode to make it out into the unstable region. This probability is exponentially suppressed in the large N limit.

5.3 Metastable plasma at high temperatures

The picture that we have found above at low temperatures and near the critical chemical potential, nicely complements the results of [11]. In [11], a careful high temperature analysis of the theory with chemical potentials on S^3 was performed. The main observations therein maybe summarized as follows. At high temperatures $TR \gg 1$, the theory behaves much like the theory on flat space. One may then derive the "electric" effective theory by integrating out all non-static fluctuations and all fluctuations on length scales of order $1/T$. This theory is valid on length scales of the order of the Debye or electric screening scale $\sim (\sqrt{\lambda}T)^{-1}$. Specifically this means that the S^3 radius $R \sim (\sqrt{\lambda}T)^{-1}$ and is parametrically smaller than

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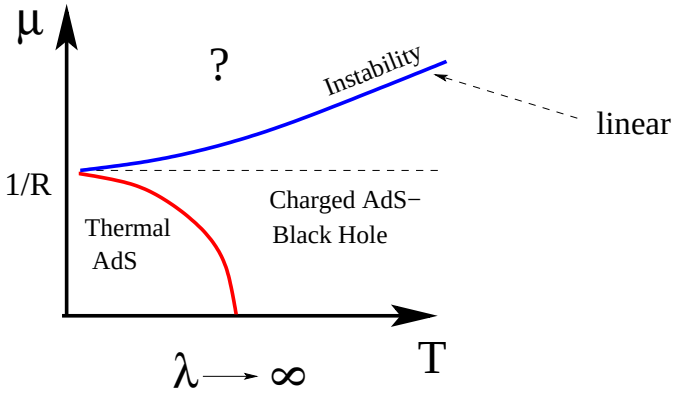

Figure 4: The phase diagram of $\mathcal{N} = 4$ theory with chemical potentials at strong coupling and infinite N. The figure corresponds to a generic pattern of chemical potentials, or angular velocities along the internal S^5 of the gravity dual. It has been argued [24] that the region above $\mu = 1/R$ and below the instability line is actually metastable in much the same way as the field theory at weak coupling.

the nonperturbative magnetic screening scale. The electric effective theory has an effective mass for the scalar fields, given by

$$m_p^2 = R^{-2} - \mu_p^2 + \lambda T^2; \qquad p = 1, 2, 3, \tag{5.16}$$

where λT^2 is the thermal or Debye mass for the scalars. This means that the effective potential for scalars will have a positive curvature at the origin, if the largest of the three chemical potentials satisfies

$$\mu_p < \sqrt{R^{-2} + \lambda T^2}. \tag{5.17}$$

At large field amplitudes, perturbative quantum corrections are expected to vanish due to large masses for the heavy states which are integrated out. In this regime the effective potential will revert to its classical behaviour. If the largest chemical potential exceeds $1/R$, then the potential has runaway behviour for large field amplitudes. Thus we are left with a metastable vacuum at the origin at high temperatures. The lifetime of this vacuum diverges exponentially at large N and as in the cases encountered in this paper, it is determined by the probability for one eigenvalue to make it out to the unstable region by thermal activation or barrier penetration. At high temperatures $TR \simeq 1/\sqrt{\lambda}$ this probablity is $\propto \exp(-N(TR)^3) \sim \exp(-N\lambda^{-3/2})$ which vanishes exponentially in the strict large N limit.

Putting together the high T results of [11] and the low T results in this article, we obtain a phase diagram for the theory at weak coupling, summarized in figure 3. (It is, of course, not known as of now whether the first order line seen in the $\lambda = 0$ theory, persists in the interacting theory.) As we discuss below, this weak coupling phase diagram bears a remarkable resemblance to the one at strong coupling (figure 4). However, there also appear to be certain intriguing and unexplained differences which we describe in some detail below.

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6. Comparison with strong coupling

In the strong coupling limit $\lambda \to \infty$, the SU(N), $\mathcal{N} = 4$ theory at infinite N, with R symmetry chemical potentials is dual to Type IIB supergravity in $AdS_5 \times S^5$ with three independent angular motions along the internal S^5 directions. Working in global coordinates, there are two possible saddle point configurations that contribute to the semiclassical gravity partition function. One of these is the "spinning" $AdS_5 \times S^5$ geometry while the other is a charged black hole in AdS. In the case where all three rotation parameters are equal (corresponding to three equal chemical potentials), the gravity dual reduces to Einstein-Maxwell theory on AdS_5 space. The system undergoes a first order Hawking-Page transition, as the temperature is increased for fixed chemical potential, from spinning thermal AdS space to the Reissner-Nordstrom-AdS black hole [30]. A similar phenomenon occurs for more general rotation parameters wherein the dual geometry is described by charged black hole solutions in 5D $\mathcal{N} = 8$ supergravity [31, 26].

As is apparent from the strong coupling phase diagram in figure 4, $\mu = R^{-1}$ is not a point of instability, except possibly at $T = 0$. Indeed, at $T = 0$, setting the chemical potential equal to R^{-1} corresponds to the internal S^5 rotating at the speed of light. Crossing this limit triggers an instability due to the time direction becoming spacelike near the origin of the space [32].

When the theory is in the charged black hole phase (deconfined plasma), increasing the chemical potential beyond $1/R$ does not lead to any local instabilities. Local instabilities are only triggered at much higher values of the chemical potential [26, 33]. The fate of the theory above this instability line remains unknown.

Figures 3 and 4 illustrate the remarkable qualitative similarities between the weakly coupled (one loop) field theory on the one hand and its strongly coupled limit on the other. In fact it has recently been argued [24] that the metastable plasma phase seen at weak coupling has a strong coupling analogue and that the charged AdS black holes are metastable for $\mu > R^{-1}$ and below the instability line. Viewing the charged black holes as near horizon limits of rotating brane configurations, the system is found to be metastable to one of the branes splitting from the system of N coincident rotating branes. This is precisely the picture expected from weak coupling arguments where the locally stable vacuum can decay by one of the scalar eigenvalues tunnelling out into the unstable region. The qualitative physical resemblance between weakly coupled gauge theory and dual gravity at strong coupling lends further support to the idea that some properties of gravity may be encoded in and extracted from weakly coupled gauge degrees of freedom.

However, while pointing out the similarities between the weak and strong coupling phase diagrams, we should also emphasize the qualitative differences. These qualitative differences are the outcome of our analysis at low temperatures in the near critical region, and continue to pose intriguing questions. Reinterpreting the phase diagram of [26] in terms of grand canonical variables, μ and T, it has been pointed out in [11, 34] that the phase diagram at strong coupling has particularly interesting features at low temperatures and near the critical chemical potential. Specifically, only when all three chemical potentials are equal $\mu_1 = \mu_2 = \mu_3 = \mu$, the black hole instability line meets the first order transition line at

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$T = 0$ and $\mu = 1/R$. At this point an extremal black hole solution with zero horizon radius becomes preferred over pure AdS. When the chemical potentials are unequal however, e.g. $\mu_2 = \mu_3 = 0$, one finds that the black hole instability line and the first order line meet at $\mu_1 = 1/R$ with $TR = 1/\pi$. At this point the charged black hole horizon shrinks to zero size and the actions of thermal AdS and charged black hole coincide. It is not known what happens to the gravity dual below this temperature. The picture we have found in weakly coupled field theory is quite different, since the instability line meets the $\mu = 1/R$ line precisely at $TR = 0$ in all cases. In all cases, at weak coupling, the instability line approaches the $\mu = 1/R$ line exponentially at low temperatures.

7. Discussion and future questions

In this paper we have investigated a corner of the phase diagram of $\mathcal{N} = 4$ theory with R symmetry chemical potentials on S^3. The region of interest corresponds to chemical potentials close to their critical values, beyond which the theory is known not to have a ground state. By explicitly computing an effective potential for the light degrees of freedom in this region, we showed that at the critical values for the chemical potentials and at zero temperature, flat directions open up. In particular, we find moduli spaces parametrized by N, $2N$ and $3N$ bosonic degrees of freedom for one, two and three critical chemical potentials respectively. These correspond to $\frac{1}{2}$ BPS, $\frac{1}{4}^{\text{th}}$ BPS and $\frac{1}{8}^{\text{th}}$ BPS sates of the $\mathcal{N} = 4$ theory, respectively becoming light at critical values for different numbers of chemical potentials. The counting of zero energy modes at the origin of the moduli spaces is consistent with this interpretation.

Using the above picture of an effective potential, we calculate the same at small temperatures at critical values of the chemical potentials. The small positive thermal masses for the light degrees of freedom allow us to move away from critical chemical potentials, and indeed to exceed the critical values, resulting in metastable ground states. In all these situations, the theory is found to be in a deconfined phase with the eigenvalues of the Polyakov loop collapsing to a point on the circle. These metastable plasma phases are the extension to low temperatures, of the metastable states at high temperature discovered in [11]. Putting together the results at low and high temperatures, the resulting phase diagram of the theory at weak coupling is summarized in figure 3. The remarkable resemblance to the strong coupling phase diagram in figure 4 is clear. However, there are crucial differences as well. The main difference to emerge from our analysis is that the width of the metastable region goes to zero only at zero temperature (exponentially). This means that the instability line meets the first order deconfinement transition line at $\mu = 1/R$ and $T = 0$ for all patterns of chemical potentials. The strong coupling picture is quite different. The only situation where the first order Hawking-Page line meets the black hole instability line at $T = 0$, is when all three chemical potentials are equal. In all other cases, the instability line and the Hawking-Page line meet at a finite temperature. Despite these differences it is quite remarkable that the metastability discovered first in weakly coupled field theory, does appear to have a strong coupling analogue [24].

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There are several related questions to pursue, along the lines of those addressed in this work. One obvious generalization of our results would be to the β-deformed theory which has three global U(1) symmetries. The free theory will have the same phase structure as the $\mathcal{N} = 4$ theory. However at non-zero weak coupling, the theories will differ. Due to lower supersymmetries, we expect that turning on different numbers of critical chemical potentials may result in scenarios quite different from the $\mathcal{N} = 4$ case. Particulary interesting is the question of the existence of a stable Bose-Einstein condensed state at weak coupling. It may be possible to look for such a phase by turning on, say more than one critical chemical potential, followed by a small temperature. Looking for such an exotic phase in the dual gravity setup would be extremely interesting.

In [12], it was pointed out that in the limit, $T \to 0$, $\mu \to R^{-1}$ with $\frac{T}{R^{-1}-\mu}$ fixed, the theory reduces to various quantum mechanical sectors, for e.g., the ferromagnetic Heisenberg spin chain. It is interesting to take this limit in our calculation of the grand canonical partition function and make contact with the results in recent studies of integrability in planar $\mathcal{N} = 4$ SYM.

Acknowledgments

We have benefitted from discussions with D. Berenstein, N. Dorey, S. Hartnoll, S. Ross, D. Son, and L. Yaffe. We thank them for their insights and comments. TJH, SPK and AN would like to thank the Isaac Newton Institute, Cambridge for hospitality during the " Strong Fields, Integrability and Strings" workshop, wherein this project received fresh impetus. SPK would like to thank the KITP, Santa Barbara for hospitality during the workshop on " Nonequilibrium dynamics in Particle Physics and Cosmology" in course of which this project was completed. This research was supported in part by the National Science Foundation under Grant no. PHY05-51164.

A. Scalar and vector Laplacians on S^3

Below we review some relevant aspects of the spherical harmonic decomposition of fields on S^3. To begin with, consider the kinetic terms for the gauge field:

$$\mathcal{L}^{(\text{gauge})} = \text{Tr}\,\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \int d^4x\,\sqrt{\det g}\text{Tr}\big(-\tfrac{1}{2}A_\mu(\tilde{D}_0^2+\Delta)A^\mu - \tfrac{1}{2}(\tilde{D}_0A_0+\nabla_iA^i)^2\big). \quad \text{(A.1)}$$

The Laplacian $\Delta = \nabla_i\nabla^i$ on S^3 depends on the tensorial nature of the object on which it acts. For example, on the vector component

$$\Delta A^i = \nabla_j\nabla^j A^i - R^i{}_j A^j\,, \quad \text{(A.2)}$$

where R_{ij} is the Ricci tensor of S^3. For the component A_0, which is a scalar on S^3, the Ricci tensor part is not present and ∇^2 is equivalent to the scalar Laplacian. The eigenvectors of the scalar Laplacian are spherical harmonics Y_ℓ labelled by angular momentum quantum numbers, $\ell \in \mathbb{Z} \geq 0$ with

$$\Delta Y_\ell = -R^{-2}\ell(\ell+2)Y_\ell \quad \text{(A.3)}$$

and degeneracy $(\ell+1)^2$. The vector field A_i can be decomposed into the image and the kernel of : $A^i = B^i + C^i$ with $\nabla_i B^i = 0$ and $C^i = \nabla^i f$. These modes have different eigenvalues for the vector Laplacian. Firstly, those in the image of ∇^i, *i.e.* C^i, are given by $\nabla^i Y_\ell$ with $\ell \in \mathbb{Z} > 0$, which satisfy

$$\Delta \nabla^i Y_\ell = -R^{-2}\ell(\ell+2)\nabla^i Y_\ell \ . \tag{A.4}$$

The remaining modes B^i, in the kernel of ∇^i, are spanned by V^i_ℓ, also labelled by the angular momentum $\ell \in \mathbb{Z} > 0$, with

$$\Delta V^i_\ell = -R^{-2}(\ell+1)^2 V^i_\ell \tag{A.5}$$

and degeneracy $2\ell(\ell+2)$. Finally, we will need the Laplacian on fermionic modes. For 2-component real fermions[8] on S^3, the fermionic Laplacian has eigenvalues $R^{-2}(\ell+\frac{1}{2})$, with $\ell \in \mathbb{Z} > 0$ and degeneracy $\ell(\ell+1)$.

The gauge field modes B_i and C_i decouple from the chemical potential and at Gaussian order the relevant terms in the Lagrangian are

$$\frac{1}{2}\mathrm{Tr}\, B_i\big[-\tilde{D}_0^2 - \Delta^{(v)} + \varphi^2\big]B^i + \frac{1}{2}\mathrm{Tr}\, B_i\big[-\tilde{D}_0^2 - \Delta^{(s)'} + \varphi^2\big]B^i\,, \tag{A.6}$$

where the superscript reminds us that the Laplacian is for divergenceless vectors and scalars, respectively. The prime means that the $\ell = 0$ mode is missing. When these fluctuations are integrated out their contribution to the effective potential is of the form

$$\frac{1}{2\beta}\sum_{ij=1}^{N}\sum_{\ell=\ell_0}^{\infty} d_\ell \log\det\big[-\tilde{D}_0^2 + \varepsilon_\ell(\varphi_{ij})^2\big]\,, \tag{A.7}$$

where d_ℓ is the degeneracy of the modes with angular momentum quantum number ℓ, so $2\ell(\ell+2)$ for B_i and $(\ell+1)^2$ for C_i. The integer ℓ_0 is the lower limit on the angular momentum quantum number, so 1 for both B_i and C_i. The quantity ε_ℓ is the energy of the mode, so equal to

$$\begin{aligned} B_i: &\qquad \varepsilon_\ell = \sqrt{R^{-2}(\ell+1)^2 + \varphi^2}\ , \\ C_i: &\qquad \varepsilon_\ell = \sqrt{R^{-2}\ell(\ell+2)^2 + \varphi^2}\ \ . \end{aligned} \tag{A.8}$$

The contributions from all the other modes have the same form for some set of $\{d_\ell, \ell_0, \varepsilon_\ell\}$.

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JHEP08(2008)046